

# Generalization of Roth's solvability criteria to systems of matrix equations\*

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## Abstract

W.E. Roth (1952) proved that the matrix equation  $AX - XB = C$  has a solution if and only if the matrices  $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$  and  $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$  are similar. A. Dmytryshyn and B. Kågström (2015) extended Roth's criterion to systems of matrix equations  $A_i X_i M_i - N_i X_i^{\sigma_i} B_i = C_i$  ( $i = 1, \dots, s$ ) with unknown matrices  $X_1, \dots, X_t$ , in which every  $X^\sigma$  is  $X$ ,  $X^\top$ , or  $X^*$ . We extend their criterion to systems of complex matrix equations that include the complex conjugation of unknown matrices. We also prove an analogous criterion for systems of quaternion matrix equations.

*AMS classification:* 15A24

*Keywords:* Systems of matrix equations, Sylvester equations, Roth's criteria

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\*Linear Algebra Appl. 527 (2017) 294–302.

# 1 Introduction

Roth [13] proved that the matrix equation  $AX - XB = C$  (respectively,  $AX - YB = C$ ) over a field has a solution if and only if the matrices  $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$  and  $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$  are similar (respectively, equivalent); see also [8, Section 4.4.22] and [10, Section 12.5].

Dmytryshyn and Kågström [4, Theorem 6.1] extended Roth's criteria to the system of generalized Sylvester equations

$$A_i X_{i'} M_i - N_i X_{i''}^{\sigma_i} B_i = C_i, \quad i = 1, \dots, s$$

with unknown matrices  $X_1, \dots, X_t$  over a field of characteristic not 2 with a fixed involution, in which every  $X_{i''}^{\sigma_i}$  is either  $X_{i''}$ , or  $X_{i''}^\top$ , or  $X_{i''}^*$ . Most of the known generalizations of Roth's criteria are special cases of their criterion. The first author was awarded the SIAM Student Paper Prize 2015 for the paper [4].

However, Dmytryshyn and Kågström [4] do not consider complex matrix equations that include the complex conjugate of unknown matrices. The theory of such equations and their applications to discrete-time antilinear systems are presented in Wu and Zhang's new book [17]. Bevis, Hall, and Hartwig [1] proved that the complex matrix equation  $A\bar{X} - XB = C$  has a solution if and only if the matrices  $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$  and  $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$  are consimilar (i.e.,  $\bar{S}^{-1} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} S = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$  for some nonsingular  $S$ ).

We extend Dmytryshyn and Kågström's criterion to a large class of matrix equations that includes the systems

$$A_i X_{i'}^{\varepsilon_i} M_i - N_i X_{i''}^{\delta_i} B_i = C_i, \quad i', i'' \in \{1, \dots, t\}, \quad i = 1, \dots, s \quad (1)$$

- of complex matrix equations, in which  $\varepsilon_i, \delta_i \in \{1, \mathbb{C}, \top, *\}$ , where  $X^{\mathbb{C}} := \bar{X}$  is the complex conjugate matrix and  $X^* := \bar{X}^\top$  is the complex adjoint matrix, and
- of quaternion matrix equations, in which  $\varepsilon_i, \delta_i \in \{1, *\}$ , where  $X^*$  is the quaternion adjoint matrix.

We prove our criterion by methods of [4] (see also [6, 15, 16]), though our exposition is self-contained and uses only elementary linear algebra.

Note that the system of matrix equations (1) over a field can be rewritten as a system  $Mx = b$  of linear equations, which gives another criterion of

solvability for (1): it has a solution if and only if  $\text{rank } M = \text{rank}[M|b]$ . However, the system  $Mx = b$  is large and can be ill-conditioned.

Special cases of the system (1) are considered in hundreds of articles and books. For recent results related to solvability criteria we refer the reader to [2, 3, 4, 5, 7, 14, 17] and the references given there. A survey of papers on Roth's criteria and their generalizations is given in the extended introduction to [7]. A quaternion linear algebra is presented in [12], in which quaternion matrix equations are considered in Chapters 5 and 14.

## 2 Main results

Let  $\mathbb{F}$  be a skew field (which can be a field). An *involution automorphism* of  $\mathbb{F}$  is a bijection  $a \mapsto a^c$  of  $\mathbb{F}$  onto itself, satisfying

$$(a + b)^c = a^c + b^c, \quad (ab)^c = a^c b^c, \quad (a^c)^c = a \quad \text{for all } a \in \mathbb{F}.$$

An *involution anti-automorphism* of  $\mathbb{F}$  is a bijection  $a \mapsto a^\circ$ , satisfying

$$(a + b)^\circ = a^\circ + b^\circ, \quad (ab)^\circ = b^\circ a^\circ, \quad (a^\circ)^\circ = a \quad \text{for all } a \in \mathbb{F}.$$

For example, the complex conjugation is an involutory automorphism and involutory anti-automorphism of  $\mathbb{C}$ ; the quaternion conjugation is an involutory anti-automorphism of  $\mathbb{H}$ .

The following theorem is proved in Section 3.

**Theorem 1.** *Given*

- *a skew field  $\mathbb{F}$  of characteristic not 2 that is finite dimensional over its center,*
- *an involutory automorphism  $a \mapsto a^c$  (possible, the identity) and an involutory anti-automorphism  $a \mapsto a^\circ$  of  $\mathbb{F}$  (possible, the identity if  $\mathbb{F}$  is a field),*
- *a system*

$$A_i X_{i'}^{\varepsilon_i} - X_{i''}^{\delta_i} B_i = C_i, \quad i = 1, \dots, s \quad (2)$$

*of matrix equations over  $\mathbb{F}$  with unknown matrices  $X_1, \dots, X_t$ , in which all  $i', i'' \in \{1, \dots, t\}$ ,  $\varepsilon_i, \delta_i \in \{1, c, \dagger, *\}$ , and*

$$A^\dagger := (A^\circ)^\top, \quad A^* := ((A^c)^\circ)^\top$$

*for each matrix  $A$  over  $\mathbb{F}$ ;*

the system (2) has a solution if and only if there exist nonsingular matrices  $P_1, \dots, P_t$  over  $\mathbb{F}$  such that

$$\begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} P_{i'}^{\langle \varepsilon_i \rangle} = P_{i''}^{\langle \delta_i \rangle} \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix}, \quad i = 1, \dots, s, \quad (3)$$

in which

$$P^{\langle \sigma \rangle} := \begin{cases} P^\sigma & \text{if } \sigma \in \{1, \mathbb{C}\}, \\ J(P^\sigma)^{-1} J^{-1} & \text{if } \sigma \in \{\dagger, *\}, \end{cases} \quad J := \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}. \quad (4)$$

If all  $\varepsilon_i, \delta_i \in \{1, \mathbb{C}\}$  in (2), then the condition “ $\mathbb{F}$  of characteristic not 2” in Theorem 1 can be omitted; see Lemma 1.

The conditions (3) on the block matrices from Theorem 1 are all given in the same style using (4). In the following remark, we give these conditions more explicitly for each of four possible cases.

*Remark 1.* For each  $i = 1, \dots, s$ , the equality (3) in Theorem 1 can be rewritten in the form:

$$\begin{aligned} \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} P_{i'}^{\varepsilon_i} &= P_{i''}^{\delta_i} \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix} && \text{if } \varepsilon_i, \delta_i \in \{1, \mathbb{C}\}, \\ P_{i''}^{\delta_i} \begin{bmatrix} 0 & -B_i \\ A_i & 0 \end{bmatrix} P_{i'}^{\varepsilon_i} &= \begin{bmatrix} 0 & -B_i \\ A_i & C_i \end{bmatrix} && \text{if } \varepsilon_i \in \{1, \mathbb{C}\}, \delta_i \in \{\dagger, *\}, \\ \begin{bmatrix} 0 & -A_i \\ B_i & 0 \end{bmatrix} &= P_{i''}^{\delta_i} \begin{bmatrix} C_i & -A_i \\ B_i & 0 \end{bmatrix} P_{i'}^{\varepsilon_i} && \text{if } \varepsilon_i \in \{\dagger, *\}, \delta_i \in \{1, \mathbb{C}\}, \\ P_{i''}^{\delta_i} \begin{bmatrix} B_i & 0 \\ 0 & A_i \end{bmatrix} &= \begin{bmatrix} B_i & 0 \\ -C_i & A_i \end{bmatrix} P_{i'}^{\varepsilon_i} && \text{if } \varepsilon_i, \delta_i \in \{\dagger, *\}. \end{aligned}$$

**Corollary 1.** (a) Over  $\mathbb{R}$ , the system (2) with  $\varepsilon_i, \delta_i \in \{1, \top\}$  has a solution if and only if (3) holds for some nonsingular real matrices  $P_1, \dots, P_t$ , and  $\top$  is used instead of  $\dagger$  in (4).

(b) Over  $\mathbb{C}$ , the system (2) with  $\varepsilon_i, \delta_i \in \{1, \mathbb{C}, \top, *\}$  has a solution if and only if (3) holds for some nonsingular complex matrices  $P_1, \dots, P_t$ . Here  $A^\mathbb{C} := \bar{A}$  is the complex conjugate matrix,  $A^* := \bar{A}^\top$  is the complex adjoint matrix. The symbol  $\top$  is used instead of  $\dagger$  in (4).

(c) Over  $\mathbb{H}$ , the system (2) with  $\varepsilon_i, \delta_i \in \{1, \mathbb{C}, \dagger, *\}$  has a solution if and only if (3) holds for some nonsingular quaternion matrices  $P_1, \dots, P_t$ . Here

$$h^\mathbb{C} := a + bi - cj - dk, \quad h^\circ := a - bi + cj + dk, \quad \bar{h} = (h^\mathbb{C})^\circ = a - bi - cj - dk$$

for each quaternion  $h = a + bi + cj + dk$ , and

$$A^\dagger = (A^\circ)^\top, \quad A^* = \bar{A}^\top$$

for each quaternion matrix  $A$ .

Note that each involutory automorphism of  $\mathbb{H}$  is either the identity, or  $h \mapsto a + bi - cj - dk$  in a suitable set of orthogonal imaginary units  $i, j, k \in \mathbb{H}$ , see [9, Lemma 1]; and each involutory anti-automorphism of  $\mathbb{H}$  is either  $h \mapsto a - bi + cj + dk$ , or  $h \mapsto a - bi - cj - dk$  in a suitable set of orthogonal imaginary units, see [12, Theorem 2.4.4(c)].

**Theorem 2.** *Let  $\mathbb{F}$  be a skew field of characteristic not 2 that is finite dimensional over its center. The system (1) over  $\mathbb{F}$ , in which all  $\varepsilon_i$  and  $\delta_i$  are as in Theorem 1, has a solution if and only if there exist nonsingular matrices  $P_1, \dots, P_t, Q_1, \dots, Q_s, R_1, \dots, R_s$  over  $\mathbb{F}$  satisfying the following 3s equalities:*

$$\left. \begin{aligned} \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} Q_i &= R_i \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix} \\ \begin{bmatrix} I & 0 \\ 0 & M_i \end{bmatrix} Q_i &= P_{i'}^{\langle \varepsilon_i \rangle} \begin{bmatrix} I & 0 \\ 0 & M_i \end{bmatrix} \\ \begin{bmatrix} N_i & 0 \\ 0 & I \end{bmatrix} P_{i''}^{\langle \delta_i \rangle} &= R_i \begin{bmatrix} N_i & 0 \\ 0 & I \end{bmatrix} \end{aligned} \right\}, \quad i = 1, \dots, s. \quad (5)$$

*Proof (assuming that Theorem 1 holds).* Define from (1) the system of 3s matrix equations

$$\left. \begin{aligned} A_i Y_i - Z_i B_i &= C_i \\ Y_i - X_{i'}^{\varepsilon_i} M_i &= 0 \\ N_i X_{i''}^{\delta_i} - Z_i &= 0 \end{aligned} \right\}, \quad i = 1, \dots, s \quad (6)$$

with unknown matrices  $X_1, \dots, X_t, Y_1, \dots, Y_s, Z_1, \dots, Z_s$ . If the system (1) has a solution  $(\underline{X}_1, \dots, \underline{X}_t)$ , then (6) has the solution  $(\underline{X}_1, \dots, \underline{X}_t; \underline{Y}_1, \dots, \underline{Y}_s; \underline{Z}_1, \dots, \underline{Z}_s)$ , in which all  $\underline{Y}_i := \underline{X}_{i'}^{\varepsilon_i} M_i$  and  $\underline{Z}_i := N_i \underline{X}_{i''}^{\delta_i}$ . Thus, the system (1) has a solution if and only if (6) has a solution. By Theorem 1, the system (6) has a solution if and only if (5) holds for some nonsingular matrices  $P_1, \dots, P_t, Q_1, \dots, Q_s, R_1, \dots, R_s$ .  $\square$

### 3 The proof of Theorem 1

The following lemma proves Theorem 1 if all  $\varepsilon_i, \delta_i \in \{1, \mathbb{C}\}$ .

**Lemma 1.** *Let  $\mathbb{F}$  be a skew field that is finite dimensional over its center. Let  $a \mapsto a^{\mathbb{C}}$  be an involutory automorphism of  $\mathbb{F}$  (which can be the identity). Let*

$$A_i X_{i'}^{\alpha_i} - X_{i''}^{\beta_i} B_i = C_i, \quad i = 1, \dots, s \quad (7)$$

*be a system of matrix equations over  $\mathbb{F}$  with unknown matrices  $X_1, \dots, X_t$ , in which all  $\alpha_i, \beta_i \in \{1, \mathbb{C}\}$ . Then the system (7) has a solution if and only if there exist nonsingular matrices  $P_1, \dots, P_t$  such that*

$$\begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} P_{i'}^{\alpha_i} = P_{i''}^{\beta_i} \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix}, \quad i = 1, \dots, s. \quad (8)$$

*Proof.*  $\implies$ . If  $(\underline{X}_1, \dots, \underline{X}_t)$  is a solution of (7), then (8) holds for

$$P_1 = \begin{bmatrix} I & \underline{X}_1 \\ 0 & I \end{bmatrix}, \dots, P_t = \begin{bmatrix} I & \underline{X}_t \\ 0 & I \end{bmatrix}. \quad (9)$$

$\impliedby$ . Suppose there are nonsingular matrices  $P_1, \dots, P_t$  of sizes  $n_1 \times n_1, \dots, n_t \times n_t$  satisfying (8). Then

$$(P_1, \dots, P_t) \in \mathcal{U} := \mathbb{F}^{n_1 \times n_1} \oplus \dots \oplus \mathbb{F}^{n_t \times n_t}. \quad (10)$$

Denote by  $C(\mathbb{F})$  the center of  $\mathbb{F}$  (which coincides with  $\mathbb{F}$  if  $\mathbb{F}$  is a field). For  $c \in C(\mathbb{F})$  and any  $a \in \mathbb{F}$ ,  $a^{\mathbb{C}} c^{\mathbb{C}} = (ac)^{\mathbb{C}} = (ca)^{\mathbb{C}} = c^{\mathbb{C}} a^{\mathbb{C}}$ , and so  $c^{\mathbb{C}} \in C(\mathbb{F})$ . Hence  $c \mapsto c^{\mathbb{C}}$  is an automorphism of  $C(\mathbb{F})$  of order 1 or 2. By [11, Chapter VI, Theorem 1.8], the index of the subfield  $\mathbb{G} := \{c \in C(\mathbb{F}) \mid c = c^{\mathbb{C}}\}$  in  $C(\mathbb{F})$  is 1 or 2. Since  $\mathbb{F}$  is finite dimensional over its center,  $\mathbb{F}$  is also finite dimensional over  $\mathbb{G}$ .

Thus, the set  $\mathcal{U}$  in (10) is a finite dimensional vector space over  $\mathbb{G}$ . Define its subspaces

$$\begin{aligned} \mathcal{U}_1 &:= \left\{ (U_1, \dots, U_t) \in \mathcal{U} \mid \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} U_{i'}^{\alpha_i} = U_{i''}^{\beta_i} \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix}, \ i = 1, \dots, s \right\}, \\ \mathcal{U}_2 &:= \left\{ (U_1, \dots, U_t) \in \mathcal{U} \mid \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} U_{i'}^{\alpha_i} = U_{i''}^{\beta_i} \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \ i = 1, \dots, s \right\}. \end{aligned}$$

Let the matrices of every

$$U = \left( \begin{bmatrix} U_{11} & U_{12} \\ U_{13} & U_{14} \end{bmatrix}, \dots, \begin{bmatrix} U_{t1} & U_{t2} \\ U_{t3} & U_{t4} \end{bmatrix} \right) \in \mathcal{U}$$

be partitioned into 4 blocks such that each  $U_{i2}$  has the same size as  $X_i$  (compare with (9)). Define the  $\mathbb{G}$ -linear mappings  $\pi_k : \mathcal{U}_k \rightarrow \mathcal{U}$  ( $k = 1, 2$ ) as follows:

$$\pi_k : \left( \begin{bmatrix} U_{11} & U_{12} \\ U_{13} & U_{14} \end{bmatrix}, \dots, \begin{bmatrix} U_{t1} & U_{t2} \\ U_{t3} & U_{t4} \end{bmatrix} \right) \mapsto \left( \begin{bmatrix} U_{11} & 0 \\ U_{13} & 0 \end{bmatrix}, \dots, \begin{bmatrix} U_{t1} & 0 \\ U_{t3} & 0 \end{bmatrix} \right).$$

Then

$$\dim_{\mathbb{G}} \text{Im } \pi_k + \dim_{\mathbb{G}} \text{Ker } \pi_k = \dim_{\mathbb{G}} \mathcal{U}_k, \quad k = 1, 2. \quad (11)$$

*Fact 1:*  $\dim_{\mathbb{G}} \mathcal{U}_1 = \dim_{\mathbb{G}} \mathcal{U}_2$ . Indeed, for the  $t$ -tuple (10) from  $\mathcal{U}_1$  and for every  $(U_1, \dots, U_t) \in \mathcal{U}_2$ , we have

$$\begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} (U_{i'} P_{i'})^{\alpha_i} = U_{i''}^{\beta_i} \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} P_{i'}^{\alpha_i} = (U_{i''} P_{i''})^{\beta_i} \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix}.$$

Hence  $(U_1, \dots, U_t) \mapsto (U_1 P_1, \dots, U_t P_t)$  is a  $\mathbb{G}$ -linear bijection  $\mathcal{U}_2 \rightarrow \mathcal{U}_1$ , which proves Fact 1.

*Fact 2:*  $\text{Ker } \pi_1 = \text{Ker } \pi_2$ . A  $t$ -tuple  $U \in \mathcal{U}$  belongs to  $\text{Ker } \pi_1$  if and only if

$$U = \left( \begin{bmatrix} 0 & U_{12} \\ 0 & U_{14} \end{bmatrix}, \dots, \begin{bmatrix} 0 & U_{t2} \\ 0 & U_{t4} \end{bmatrix} \right) \in \mathcal{U}_1$$

if and only if

$$U = \left( \begin{bmatrix} 0 & U_{12} \\ 0 & U_{14} \end{bmatrix}, \dots, \begin{bmatrix} 0 & U_{t2} \\ 0 & U_{t4} \end{bmatrix} \right) \in \mathcal{U}_2$$

if and only if  $U \in \text{Ker } \pi_2$ .

*Fact 3:*  $\text{Im } \pi_1 \subset \text{Im } \pi_2$ . For each

$$U = \left( \begin{bmatrix} U_{11} & 0 \\ U_{13} & 0 \end{bmatrix}, \dots, \begin{bmatrix} U_{t1} & 0 \\ U_{t3} & 0 \end{bmatrix} \right) \in \text{Im } \pi_1,$$

there exist  $U_{12}, U_{14}, \dots, U_{t2}, U_{t4}$  such that

$$\left( \begin{bmatrix} U_{11} & U_{12} \\ U_{13} & U_{14} \end{bmatrix}, \dots, \begin{bmatrix} U_{t1} & U_{t2} \\ U_{t3} & U_{t4} \end{bmatrix} \right) \in \mathcal{U}_1,$$

which means that

$$\begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} \begin{bmatrix} U_{i'1}^{\alpha_i} & U_{i'2}^{\alpha_i} \\ U_{i'3}^{\alpha_i} & U_{i'4}^{\alpha_i} \end{bmatrix} = \begin{bmatrix} U_{i''1}^{\beta_i} & U_{i''2}^{\beta_i} \\ U_{i''3}^{\beta_i} & U_{i''4}^{\beta_i} \end{bmatrix} \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix}, \quad i = 1, \dots, s.$$

Then

$$\begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} \begin{bmatrix} U_{i'1}^{\alpha_i} & 0 \\ U_{i'3}^{\alpha_i} & 0 \end{bmatrix} = \begin{bmatrix} U_{i''1}^{\beta_i} & 0 \\ U_{i''3}^{\beta_i} & 0 \end{bmatrix} \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}, \quad i = 1, \dots, s,$$

and so  $U \in \text{Im } \pi_2$ , which proves Fact 3.

By (11) and Facts 1–3,  $\text{Im } \pi_1 = \text{Im } \pi_2$ . Since  $(I, \dots, I) \in \mathcal{U}_2$ ,  $(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \dots, \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}) \in \text{Im } \pi_2 = \text{Im } \pi_1$ . Hence there are  $U_{12}, U_{14}, \dots, U_{t2}, U_{t4}$  such that

$$\left( \begin{bmatrix} I & U_{12} \\ 0 & U_{14} \end{bmatrix}, \dots, \begin{bmatrix} I & U_{t2} \\ 0 & U_{t4} \end{bmatrix} \right) \in \mathcal{U}_1,$$

which means that

$$\begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} \begin{bmatrix} I & U_{i'2}^{\alpha_i} \\ 0 & U_{i'4}^{\alpha_i} \end{bmatrix} = \begin{bmatrix} I & U_{i''2}^{\beta_i} \\ 0 & U_{i''4}^{\beta_i} \end{bmatrix} \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix}, \quad i = 1, \dots, s. \quad (12)$$

Equating the (1,2) blocks in (12), we get  $A_i U_{i'2}^{\alpha_i} = C_i + U_{i''2}^{\beta_i} B_i$ . Thus,  $(U_{12}, \dots, U_{t2})$  is a solution of the system (7).  $\square$

*Proof of Theorem 1.*  $\implies$ . If  $(\underline{X}_1, \dots, \underline{X}_t)$  is a solution of (2), then the equalities (3) hold for  $P_1, \dots, P_t$  defined in (9).

$\impliedby$ . Suppose there are nonsingular matrices  $P_1, \dots, P_t$  satisfying (3). We consider the set  $\{1, \mathbb{C}, \dagger, *\}$  as the abelian group with multiplication

	1	$\mathbb{C}$	$\dagger$	$*$
1	1	$\mathbb{C}$	$\dagger$	$*$
$\mathbb{C}$	$\mathbb{C}$	1	$*$	$\dagger$
$\dagger$	$\dagger$	$*$	1	$\mathbb{C}$
$*$	$*$	$\dagger$	$\mathbb{C}$	1

that corresponds to the compositions of the matrix mappings  $A \mapsto A^\varepsilon$ ,  $\varepsilon \in \{1, \mathbb{C}, \dagger, *\}$ .

Represent (3) in the form

$$\begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix} \left( P_{i'}^{\langle \lambda_i \rangle} \right)^{\alpha_i} = \left( P_{i''}^{\langle \mu_i \rangle} \right)^{\beta_i} \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix}, \quad i = 1, \dots, s, \quad (13)$$



in which  $\alpha_i, \beta_i \in \{1, \mathbb{C}\}$  and  $\lambda_i, \mu_i \in \{1, \dagger\}$  are such that  $\alpha_i \lambda_i = \varepsilon_i$  and  $\beta_i \mu_i = \delta_i$ . Applying  $\dagger$  to (13) and multiplying each factor by  $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$  on the left and by  $J^{-1} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$  on the right, we get

$$J \left( P_{i'}^{\langle \alpha_i \lambda_i \rangle} \right)^\dagger J^{-1} J \begin{bmatrix} A_i & 0 \\ 0 & B_i \end{bmatrix}^\dagger J^{-1} = J \begin{bmatrix} A_i & C_i \\ 0 & B_i \end{bmatrix}^\dagger J^{-1} J \left( P_{i''}^{\langle \beta_i \mu_i \rangle} \right)^\dagger J^{-1}. \quad (14)$$

Using

$$\begin{aligned} J \left( \left( P_{i'}^{\langle \alpha_i \lambda_i \rangle} \right)^\dagger \right)^{-1} J^{-1} &= \left( P_{i'}^{\langle \alpha_i \lambda_i \rangle} \right)^{\langle \dagger \rangle} = \left( P_{i'}^{\langle \lambda_i \dagger \rangle} \right)^{\alpha_i}, \\ J \left( \left( P_{i''}^{\langle \beta_i \mu_i \rangle} \right)^\dagger \right)^{-1} J^{-1} &= \left( P_{i''}^{\langle \beta_i \mu_i \rangle} \right)^{\langle \dagger \rangle} = \left( P_{i''}^{\langle \mu_i \dagger \rangle} \right)^{\beta_i} \end{aligned}$$

and (4), we rewrite (14) as follows:

$$\begin{bmatrix} B_i^\dagger & 0 \\ 0 & A_i^\dagger \end{bmatrix} \left( P_{i''}^{\langle \mu_i \dagger \rangle} \right)^{\beta_i} = \left( P_{i'}^{\langle \lambda_i \dagger \rangle} \right)^{\alpha_i} \begin{bmatrix} B_i^\dagger & -C_i^\dagger \\ 0 & A_i^\dagger \end{bmatrix}, \quad i = 1, \dots, s. \quad (15)$$

The equalities (13) and (15) and Lemma 1 ensure the solvability of the system formed by  $2s$  matrix equations

$$A_i Y_{\lambda_i, i'}^{\alpha_i} - Y_{\mu_i, i''}^{\beta_i} B_i = C_i, \quad B_i^\dagger Y_{\mu_i \dagger, i''}^{\beta_i} - Y_{\lambda_i \dagger, i'}^{\alpha_i} A_i^\dagger = -C_i^\dagger \quad (16)$$

( $i = 1, \dots, s$ ) with unknown matrices  $Y_{1,1}, \dots, Y_{1,t}, Y_{\dagger,1}, \dots, Y_{\dagger,t}$ . Let  $\underline{Y}_{1,1}, \dots, \underline{Y}_{1,t}, \underline{Y}_{\dagger,1}, \dots, \underline{Y}_{\dagger,t}$  be its solution. Substituting these matrices to (16) and applying  $\dagger$  to the right equalities, we get

$$A_i \underline{Y}_{\lambda_i, i'}^{\alpha_i} - \underline{Y}_{\mu_i, i''}^{\beta_i} B_i = C_i, \quad A_i \left( \underline{Y}_{\lambda_i \dagger, i'}^{\alpha_i} \right)^\dagger - \left( \underline{Y}_{\mu_i \dagger, i''}^{\beta_i} \right)^\dagger B_i = C_i.$$

Adding the left and right equalities, we obtain

$$A_i \left( \underline{Y}_{\lambda_i, i'} + \underline{Y}_{\lambda_i \dagger, i'}^\dagger \right)^{\alpha_i} - \left( \underline{Y}_{\mu_i, i''} + \underline{Y}_{\mu_i \dagger, i''}^\dagger \right)^{\beta_i} B_i = 2C_i, \quad i = 1, \dots, s. \quad (17)$$

Write  $\underline{X}_i := (\underline{Y}_{1,i} + \underline{Y}_{\dagger,i}^\dagger)/2$  for  $i = 1, \dots, t$ . Then  $\underline{X}_i^\dagger = (\underline{Y}_{\dagger,i} + \underline{Y}_{1,i}^\dagger)/2$ . By (17),

$$A_i \underline{X}_{i'}^{\alpha_i \lambda_i} - \underline{X}_{i''}^{\beta_i \mu_i} B_i = C_i, \quad i = 1, \dots, s.$$

Therefore,  $\underline{X}_1, \dots, \underline{X}_t$  is a solution of the system (2).  $\square$

## Acknowledgements

A. Dmytryshyn was supported by the Swedish Research Council (VR) grant E0485301, and by eSSENCE, a strategic collaborative e-Science programme funded by the Swedish Research Council. V. Futorny was supported by CNPq grant 301320/2013-6 and FAPESP grant 2014/09310-5. V.V. Sergeichuk was supported by FAPESP grant 2015/05864-9.

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